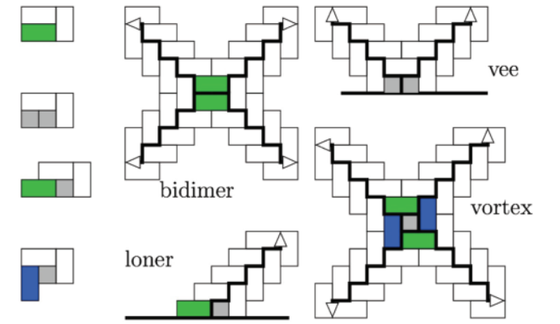
Year 10
Key Stage 4

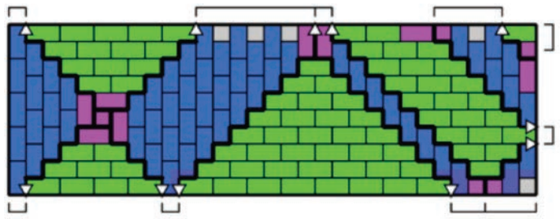
HOW TO MAKE A MATHEMATICAL DISCOVERY: THE TATAMI RESTRICTION AT LARGE

K.Hancock, supervised by A. Erickson

Abstract

The Scholars Programme course *How to Make a Mathematical Discovery* guides pupils through the process of discovering and proving mathematical theorems in the context of locally restricted arrangements of tiles. After studying restricted arrangements of 1x1 monomino tiles and 1x2 domino tiles on rectangular grids, pupils apply the methods they have learnt in order to make mathematical discoveries with a different set of tiles and restrictions, collectively called tatami *restrictions*. They are asked to give a brief evaluation of various tatami restrictions, and then to narrate any discoveries that they have made about the 5-tatami restriction for arrangements of triangles and lozenges on the isometric grid. The two figures from the course material shown below are central to this Scholars Programme and provide some context for the article. The first is a summary of significant tile-arrangements that can occur on rectangular-grids, and the second is an example of monomino-domino tatami covering in which no four tiles meet at any point, and that uses all of the significant arrangements (up to rotation and reflection) shown in the first image.





Section 1 Introduction

We consider restricted arrangements of triangle and lozenge tiles on isometric grids. A tatami restriction is a restriction where certain numbers of tiles cannot meet at one point on a grid. This restriction can range from 2-tatami (where two tiles cannot meet) up to 6-tatami (where six tiles cannot meet) on an isometric grid. Fundamentally, the study of tatami coverings is discrete rather than continuous. The tatami-restriction creates 'forced' placements of tiles, so that the tiles do not violate or conflict with the restriction itself.

Section 2 Discussion of various tatami restrictions

Section 2.1 The 2-tatami restriction

The 2-tatami restriction is satisfied when two tiles do not meet anywhere on an isometric grid. This creates a pattern that is combinatorially uninteresting because it is too restrictive, as the tiles have to be isolated, so they do not meet at a point. Therefore, the isometric grid cannot be completed with this restriction because two tiles will meet at a point if they are placed beside each other. However, small grids can be covered with the 2-tatami restriction; in particular, a tile can be placed on a grid with space for one tile, so that the grid is completed without opposing the restriction.

Section 2.2 The 3-tatami restriction

The 3-tatami restriction is satisfied when three tiles do not meet anywhere on the isometric grid. Moreover, I think that this restriction creates a combinatorially uninteresting pattern because most large grids cannot be completed without creating a pattern where three or more tiles meet at a point. This is because there can only be two tiles meeting at a point in order to abide by the restriction and there would have to be four tiles meeting at a point in order to complete the isometric grid. Consequently, this restriction is too restrictive due to the fact that the whole isometric grid cannot be completely covered.

Alternatively, the grid can be completed if it has only one row because one row of tiles can be placed on such a grid so that only two tiles are meeting at a point. Thus, the restriction has not been violated; this is displayed in Fig. 1. The 3-tatami restriction is not as restrictive as the 2-tatami restriction because grids with only a single row can be completed.

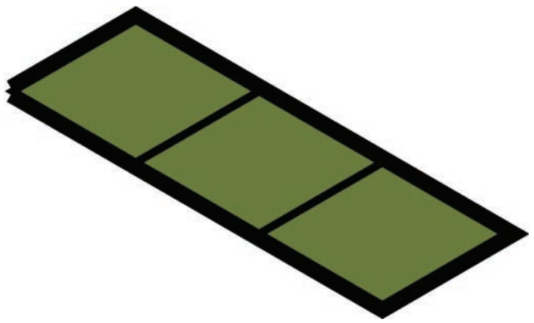


Fig. 1.

Section 2.3 The 4-tatami restriction

The 4-tatami restriction forbids configurations of tiles that have the following property: on the vertices of the isometric grid, four tiles meet at a point. Therefore, similar to the 3-tatami restriction, it is also quite restrictive, because certain large grids cannot be completed. For example, rays, an important structure in 5-tatami coverings (such as in Fig. 7), cannot be constructed without 4-tiles meeting at a point, so the grid is unable to be completed without opposing the restriction. This means that the tiles can only be placed on the grid when they are partially isolated from each other or they meet at a point with three or fewer tiles (such as in the centre of Fig. 7). However, they cannot be placed with four tiles meeting to complete the grid, as a violation will take place.

Section 2.4 The 5-tatami restriction

The 5-tatami restriction is combinatorially interesting because it allows a wide variety of patterns to be created without a violation of the restriction taking place. In addition, rays, shown in Fig. 6 and described in Erickson (2013), can be created by the continuation of certain configurations of four tiles meeting at a point, so that the isometric grid can be completed. Some of the tiles can then be flipped so that the ray has changed, but the violation has still not occurred. Therefore, I think that the 5-tatami restriction is the most interesting out of the restrictions I have discovered, because it is a restriction that allows many ideas to be fulfilled without conflicting with the restriction.

The idea of flipping a row is demonstrated via the coloured tiles in Fig. 2. Different rows from this isometric grid can be flipped, like the row shown, without the restriction being violated. A violation does not take place because only half of the ray is being flipped, so there is still a continuous ray that is similar to the beginning ray. This causes, in my opinion, the most fascinating restriction, because it is a catalyst for many designs to be created with different amounts of lozenge and triangle tiles included in the isometric grid. Additionally, in most cases the triangle tiles can only be placed on the outer rows of the isometric grid in order to abide by the tatami restriction, because there will be 5 or 6 tiles meeting at a point if the lozenge is placed away from the edge of the isometric grid. The triangle tile creates placements that are 'forced' and these placements create a circular design, so five or six tiles meet at a point, and this violates the 5-tatami restriction.

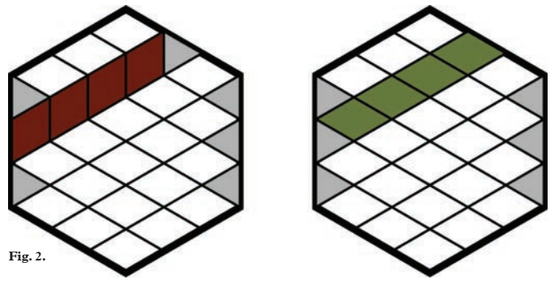


Fig. 2.

Section 2.4 The 6-tatami restriction

The 6-tatami restriction allows the most variety of patterns to be created without violating the restriction, as the only pattern that violates this restriction looks similar to a flower, with the ends of six tiles meeting at one point, as in Fig. 3. Moreover, this restriction is also combinatorially interesting because different designs can be created with a wide variety of patterns. Many such patterns that would otherwise violate the 2, 3, 4, and 5-tatami restrictions can be placed on the grid without violating the 6-tatami restriction. Therefore, this restriction is rather unrestrictive because almost every covering is possible. It can therefore also be argued that this makes the 6-tatami an uninteresting restriction, because almost any design is possible.

Section 3 Further Observations and discoveries

With the wording of the tatami restrictions, I began to wonder whether you could place four, five, or six tiles meeting at a point without violating the 3-tatami restriction. This is because the 3-tatami restriction only stated that three tiles could not meet at a point; it never specifically stated that four, five, or six tiles could not meet at a point because it would violate the restriction. Consequently, I began to draw some of the 4, 5, and 6-tatami violations to observe whether the restriction would be violated. After further observations, I realised that three tiles also meet at a point where four, five, or six tiles meet and that the 3-tatami restriction would probably also be violated; this can be visualised in Fig. 3. The coloured tiles represent three tiles that meet at a point on these violations. Therefore, the 3-tatami restriction is violated where four, five, or six tiles meet at a point. This illustrates the fact that a maximum of two tiles can be placed next to each other in order to abide by the restriction.

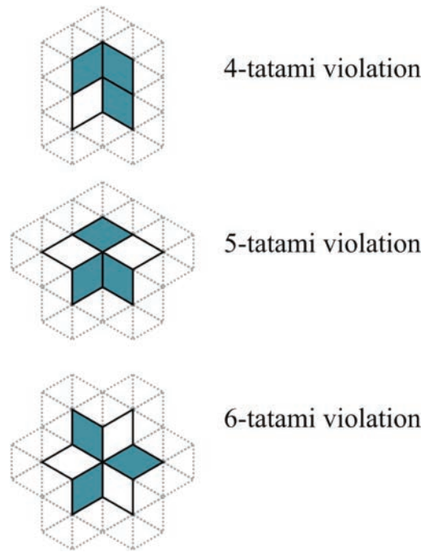


Fig. 3.

Theorem

The 5-tatami restriction includes a pattern that resembles a sodium chloride crystal.

Proof

The structure of a 5-tatami arrangement includes rays and has an organised appearance, such as where 4 lozenge tiles are placed at one point together. Placing the tatami tiles in an interesting way can create the illusion of sodium chloride crystal-like cube. This is demonstrated in Fig. 4.

The structure of this 5-tatami arrangement consists purely of diamonds, and they are placed so that four tiles meet at one point. These 4-tile arrangements are then placed in rows that are then rotated to fit inside the isometric grid. Moreover, this abides by the 5-tatami restriction because at most four tiles meet at each point of the grid and it also creates an illusion of a cube. Furthermore, this pattern is interesting despite the fact that it is simple and organised. This is combinatorially interesting because this pattern illustrates a local rule for the placement of individual parts needed to build something complex like a sodium chloride crystal.

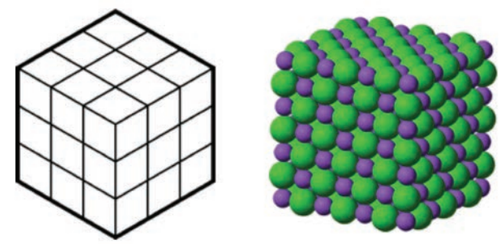


Fig. 4 and fig. 5. Sodium chloride crystal

Interestingly, the structure shown in Fig. 4 resembles the structure of a sodium chloride crystal, shown in Fig. 5. This is because the particles inside the crystal are fixed in rows and columns and the tiles in the pattern are also organised in rows that rotate, so they complete the isometric grid covering. The global structure of the 5-tatami restriction, in a way, echoes the structure of a sodium chloride crystal, perhaps because of a resemblance between the local rules of these structures; a local rule that emphasises the

strong relation between the tiles or particles. The tiles or particles are locally organised in a specific, orderly fashion to create

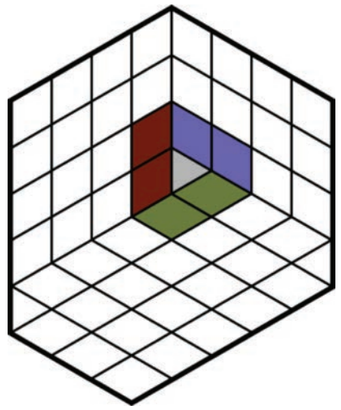


Fig. 6.

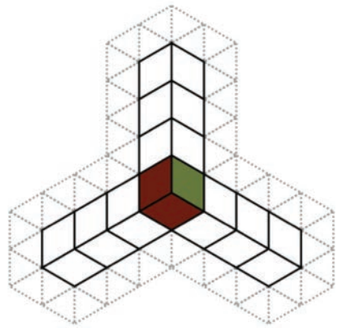


Fig. 7.

Section 3.1 The vortex

After studying the four essential configurations, loners, vees, bidimers, and vortices, for the beginning of a ray on a square grid with monominos and dominos, defined in Erickson (2013), I began to test whether the rays would begin the same way on an isometric grid. I realised that a vortex could be used to start a ray on an isometric grid, where the starting point is a triangle and the surrounding ray is made up of lozenges. Furthermore, I based this discovery on the fact that the square-grid vortex starts with a monomino on a grid based upon monominos and dominos, so I realised that a vortex on the isometric grid would have to begin with a triangle surrounded by lozenges. This will mean that a grid can be successfully completed. Surprisingly, the vortex also abides by the 5-tatami restriction, so the 5-tatami restriction is, in my opinion, extremely interesting due to all the possible tatami coverings that can be completed without a violation. This means that the 5 tatami restriction is not very restrictive and it allows us to think about all the possible ways in which the tiles can be placed in order to create coverings that match the restriction itself. The configuration of a vortex on an isometric grid is shown in Fig. 6.

The coloured section demonstrates the beginning of the vortex rays, with the triangle surrounded by lozenges. As you can see, this is very similar to the vortex created on a grid with monominos and dominos (Editor's note: The configuration referred to is provided in the abstract). The properties of both of these grids are therefore very similar, and the configuration of the vortex can be adapted to the isometric grid.

Section 3.1 The triple bidimer

As the vortex could successfully be completed, I started to investigate an isometric way of creating a bidimer. I did this by placing two diamonds together like a bidimer and then beginning to create rays from the two diamonds. There could only be three rays created from the centre of the triple bidimer because of the isometric grid restricting the triple bidimer to three rays. This is due to the fact that the placement of another diamond is forced in the centre of the triple bidimer in order for rays to be created. Therefore, there are only three corners in which rays can be developed from, rather than four corners like in a rectangular grid. My idea is highlighted in Fig. 7.

The colour red illustrates my initial idea to create two lozenges and place them together like a bidimer on a rectangular grid, but I had to adapt my initial idea to fit the isometric grid. This is demonstrated by the addition of another diamond, which is green, to create a shape that is almost like a cube in the centre of the triple bidimer. As a result of the addition of a diamond, there could only be three rays developed from the corners of the starting shape. This is adaptation is based upon the bidimer that was created for the rectangular grid (Editor's note: The configuration referred to is provided in the abstract).

Section 4 Conclusion

In conclusion, the tatami restricted arrangements of tiles are in general a combinatorially interesting concept that frames many ideas and discoveries. Tatami restrictions range from being too restrictive to not very restrictive at all and that makes tatami, in my opinion, very interesting and a fascinating concept to investigate and make new discoveries on. The local rules that I have been investigating contribute to complex structures on a larger scale; for example, I learnt that most (unrestricted) arrangements of lozenge coverings form an 'Arctic circle'[1].

Tatami is the most fascinating part of mathematics that I have developed an understanding of so far, as the restrictions are almost like a foundation to discover many different things through existing objects and through mathematical prospects, like the 'Arctic circle'.

Bibliography

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About the authors

K. is a Year 10 pupil at The Academy at Shotton Hall in Peterlee, County Durham. A. Erickson completed his PhD on discrete mathematics in 2013 in The Department of Computer Science at The University of Victoria in Victoria, Canada.

PhD Tutor's note

K. Hancock's mathematical maturity far exceeds the expected level of her age group and the narration of her mathematical discoveries demonstrates a keen interest in the subject and intuitive understanding of the tatami arrangements. One of the challenges encountered in this subject area is the communication of complex ideas with precision; in the article, K. Hancock describes complex mathematical structures by combining diagrams, making comparisons to familiar objects, and effectively employing a wide range of descriptive vocabulary, including many words that were introduced in the Scholars Programme.

Year 10
Key Stage 4

DESCRIBE ALZHEIMER'S, PARKINSON'S AND MOTOR NEURONE DISEASE AND COMPARE AND CONTRAST EACH CONDITION. DESCRIBE AND EVALUATE THE PROS AND CONS OF ONE OF THE NOVEL THERAPIES FOR THESE DISEASES.

M. Hui Qi Tan, supervised by P. Smethurst

Abstract

As the average age of the population increases so does the prevalence of neurodegenerative diseases. The most common forms these neurodegenerative diseases take are Alzheimer's, Parkinson's and Motor Neuron Disease which all affect the brain and/or spinal cord, and have devastating implications for the individual and family affected. These diseases are putting the economy and public health services under severe strain due to the lack of effective treatments and high costs of palliative care. We still have a long way to go to fill in the large gaps in our knowledge of these conditions and we desperately require effective treatments to alter the disease and improve the patient's quality of life. This essay will describe what is known about these conditions including: what the condition is, what the symptoms are, what some of the pathological features are and finally what treatments are available. Also covered here are the similarities and differences between these disorders, what stem cell therapy is and how it has potential use for modelling and treating these diseases. The ultimate aim here is to highlight the need for action and further research to investigate how these diseases work in order to develop effective disease models and treatments.

Introduction

With millions of people affected worldwide, neurodegenerative diseases have become much more prevalent in our current society, leading to researchers investigating ways to treat these ubiquitous diseases of the brain. Ranging from Alzheimer's Disease (AD) to Parkinson's Disease (PD) and forms of Motor Neurone Disease such as Amyotrophic Lateral Sclerosis (ALS), neurodegeneration or the death of neurones in the central nervous system are still under thorough research as to how these diseases affect people, and more significantly, what causes neurodegeneration itself. Despite the treatments and therapeutic methods available for other disorders in the world such as diabetes, there is still no finalised cure for any of the three major neurological diseases. 'Neurological disorders remain neglected and ignored…'[1] – their complexity in terms of their causes and complication in ways of treating the diseases mean that these heterogeneous disorders are still the cause of 4% of all deaths worldwide.[2] However, with the drastic improvement in medical technology – for example, the use of positron emission tomography (PET) – researchers are beginning to explore new novel treatments such as stem cell therapy, which creates hope that someday there will be an ultimate cure for all of the neurodegenerative diseases we are faced with.

Alzheimer's Disease

Alzheimer's Disease (AD) is a chronic neurodegenerative disease affecting over 40 million people worldwide, with the common symptoms being cognitive dysfunction and confusion. The progressive and incurable quality of this disease results in AD being the sixth greatest cause of all deaths in the United States alone.[3] Much like other neurodegenerative diseases such as Amyotrophic Lateral Sclerosis (ALS) – a common form of Motor Neurone Disease – AD is also linked to ageing, although other risk factors have also been believed to be a cause of the disease. Similarly to Parkinson's disease, one of the proposed factors linked to AD is the decrease in levels of a certain neurotransmitter that is essential in regulating specific bodily systems. In AD, it is believed that the decrease in the levels of acetylcholine (ACh) neurotransmitters found in cholinergic neurones in some parts of the central nervous system (CNS) are one of the causes of confusion and cognitive dysfunction – symptoms are evident amongst 20% of AD patients.[4] As acetylcholine is vital in the transfer of electrochemical impulses between cholinergic neurones that enable cognitive processing, the decrease in this particular neurotransmitter means that cognitive ability is impaired. The progression of AD (with the progressive decrease in ACh) links to the deterioration in AD patients' cognitive ability. The scores for mild AD are nearly three times greater than that of severe Alzheimer's disease, reiterating how the lack of the acetylcholine neurotransmitter causes symptoms of AD to worsen over time.

A major cause associated with Alzheimer's Disease is the amyloid plaques between neurones in AD patients. The formations of these amyloid plaques are from parts of an amyloid precursor protein (APP), called beta amyloid, which are congested between nerve cells in the brain.[5] The insoluble plaques disrupt nerve signalling and the delivery of electrochemical impulses to neurones as synaptic transmission is prolonged. Both AD and PD involve the abnormality of certain protein structures – however, in PD, it is the abnormality in alpha-synuclein and its function that causes symptoms to occur. In terms of AD, another major cause of this disease is the neurofibrillary tangles in the brain that are a result of the abnormal structure of Tau, a protein that stabilises structures called microtubules which are fundamental for the transportation of nutrients between neurones. The microtubules are also essential for the signalling system between nerve cells. In a person without AD, Tau molecules bind to microtubules to form the necessary structures; in a person with AD, on the other hand, Tau molecules connect with more molecules of Tau which cause neurofibrillary tangles to develop within the neurones. This causes the degeneration of neurones in the brain as the neurofibrillary tangles disintegrate the microtubules. Indeed, the lack of nutrients being delivered to the neurones and the additional collapse of neuronal systems that control the transmission of signals can also cause neurodegeneration.[6]

The main difference that sets AD apart from PD and ALS is the area in the brain in which the neurological disease affects. The symptoms of each disease have a direct correlation to the parts of the brain or body that are affected. In AD, affected areas include the temporal lobes and the hippocampus in the brain; in PD, a region in the mid brain called the substantia nigra is affected; lastly, in ALS, the nerves and the muscles – in particular, the neurones extending from the brain to the spinal cord of the CNS – are affected.[7] In terms of AD, for instance, as elucidated in Fig. 2, brain atrophy (the shrinkage of the brain) is present as the whole Alzheimer's brain is significantly smaller in size in comparison with the normal brain. Researches have supported the findings of brain atrophy in AD patients, as they found that AD brains are 10% smaller in mass: the average adult brain being 1.3-1.4kg; an AD brain being 1.17-1.26kg.[8] Fig. 1 also depicts how the loss of glucose uptake (shown in the Positron Emission Tomography scan) in the upper and lateral sections of the brain have caused symptoms associated with those areas to occur. For instance, the frontal cortex and the temporal lobes, where the amydala and hippocampus are located, are areas of low glucose uptake. This helps to explain how AD patients have difficulty with learning and memory (as the hippocampus has been affected) and even speech production and word recognition since the Wernicke's area (in control of word recognition) and the Broca's area (in control of speech production) are located in the affected temporal lobes. The darker areas and gaps in the image of the AD brain also indicate enlarged ventricles, helping to explain how patients with AD have much more severe cognitive impairment (60% more severe) in comparison with those with milder cognitive impairment.[9]